On the Chermak-Delgado lattice of a finite group

Ryan McCulloch and Marius Tărnăuceanu

May 30, 2019

Abstract

By imposing conditions upon the index of a self-centralizing subgroup of a group, and upon the index of the center of the group, we are able to classify the Chermak-Delgado lattice of the group. This is our main result. We use this result to classify the Chermak-Delgado lattices of dicyclic groups and of metabelian p-groups of maximal class.

MSC2000 : Primary 20D30; Secondary 20D60, 20D99.
Key words : Chermak-Delgado measure, Chermak-Delgado lattice, Chermak-Delgado subgroup, subgroup lattice, (generalized) dicyclic group, metabelian group.

1 Introduction

Throughout this paper, G will denote a finite group. Denote by

$$m_G(H) = |H||C_G(H)|$$

the *Chermak-Delgado measure* of a subgroup H of G and let

$$m^*(G) = \max\{m_G(H) \mid H \leq G\} \text{ and } \mathcal{CD}(G) = \{H \leq G \mid m_G(H) = m^*(G)\}.$$

The set $\mathcal{CD}(G)$ forms a modular, self-dual sublattice of the lattice of subgroups of G, which is called the *Chermak-Delgado lattice* of G. It was first introduced by Chermak and Delgado [5], and revisited by Isaacs [7]. In the last years there has been a growing interest in understanding this lattice (see e.g. [1, 2, 3, 4, 6, 8, 9, 10, 13, 14, 15, 18]). Recall two important properties of the Chermak-Delgado lattice that will be used in our paper:

· if $H \in \mathcal{CD}(G)$, then $C_G(H) \in \mathcal{CD}(G)$ and $C_G(C_G(H)) = H$;

· the minimum subgroup $M(G)$ of $\mathcal{CD}(G)$ (called the *Chermak-Delgado subgroup* of G) is characteristic, abelian, and contains $Z(G)$.

For a positive integer $n \geq 1$, the dicyclic group of order $4n$, usually denoted by Dic_{4n}, is defined as

$$Dic_{4n} = \langle a, x \mid a^{2n} = 1, x^2 = a^n, a^x = a^{-1} \rangle.$$

This has the following generalization: given an arbitrary abelian group A of order $2n$, the generalized dicyclic group induced by A is defined as

$$Dic_{4n}(A) = \langle A, x \mid x^4 = 1, x^2 \in A \setminus \{1\}, a^x = a^{-1}, \forall\, a \in A \rangle.$$

Obviously, we have $Dic_{4n}(\mathbb{Z}_{2n}) = Dic_{4n}$. It is also easy to see that if $\exp(A) = 2$, that is that A is an elementary abelian 2-group, then $Dic_{4n}(A)$ is abelian and consequently $\mathcal{CD}(Dic_{4n}(A)) = \{Dic_{4n}(A)\}$. Note that if $\exp(A) \neq 2$, then

$$Z(Dic_{4n}(A)) = \{a \in A \mid a^2 = 1\} \cong \frac{A}{A^2}\,,$$

where $A^2 = \{a^2 \mid a \in A\}$.

A finite group G is said to be *metabelian* if the derived subgroup, G', is abelian. Equivalently, a finite group G is metabelian if there exists an abelian normal subgroup A of G so that G/A is abelian.

A finite p-group G of order p^n is said to be *of maximal class* if the nilpotence class of G is $n-1$. The following results on p-groups will be useful to us. Lemma 1.1 appears in (4.26), [12], II, Lemma 1.2 appears in Theorem 2.4 of [16], and Lemma 1.3 at end of [14].

Lemma 1.1. *Any group of order p^4 contains an abelian subgroup of order p^3.*

Lemma 1.2. *Suppose G is a p-group of order p^n and G is of maximal class. Then $|Z(G)| = p$, $|G : G'| = p^2$, and for each $2 \leq i \leq n$, we have that G_i is the unique normal subgroup of G order p^{n-i}, where G_i is the ith term in the lower central series for G.*

Lemma 1.3. *Let G be a p-group of maximal class and of order p^5. If $m^*(G) = p^6$, then $\mathcal{CD}(G) = \{G, T, G', A_1, \ldots, A_p, Z(T), Z(G)\}$, where $|T| = p^4$, $|G'| = |A_i| = p^3$ for each i, $|Z(T)| = p^2$, and $G', A_1, \ldots, A_p$ are all abelian and distinct. Also, none of $A_1, \ldots, A_p$ are normal in G.*

2 Main Results

In this section we present a result which generalizes Proposition 7 in [10]. It will be used both in Sections 3 and 4. A subgroup A of G is said to be self-centralizing if $C_G(A) = A$. An important observation used here is that if $T \in \mathcal{CD}(G)$, then since $Z(G) \leq T$, we have that $|G : T|$ divides $|G : Z(G)|$. So by imposing conditions on $|G : Z(G)|$, we are able to obtain results about $\mathcal{CD}(G)$.

Theorem 2.1. *Let G be a finite group, let p be a prime, and among the self-centralizing subgroups of G, let A be one of maximum order.*

1) *If G is abelian, then $\mathcal{CD}(G) = \{G\}$.*

2) *If $|G : Z(G)| = p^2$, then $\mathcal{CD}(G) = \{Z(G), A, A_1, \ldots, A_p, G\}$ is a quasi-antichain of width $p + 1$, with each A_i abelian.*

3) *If $|G : A| = p$ and $|G : Z(G)| = p^i$ with $i > 2$, then $\mathcal{CD}(G) = \{A\}$.*

   If p is the smallest prime divisor of $|G|$ and $|G : A| = p$ and $|G : Z(G)| > p^2$, then $\mathcal{CD}(G) = \{A\}$.

4) *Suppose $|G : A| = p^2$.*

   a) *If $|G : Z(G)| = p^3$, then $\mathcal{CD}(G) = \{Z(G), G\}$.*
   If p is the smallest prime divisor of $|G|$ and $p^2 < |G : Z(G)| < p^4$, then $\mathcal{CD}(G) = \{Z(G), G\}$.

   b) *If $|G : Z(G)| = p^4$, then*

   $$\mathcal{CD}(G) = \{Z(G), Z(T_1), \ldots, Z(T_n), A, A_1, \ldots, A_m, T_1, \ldots, T_n, G\}$$

   where $T_1, \ldots, T_n$ $(n \geq 0)$ are all of the subgroups of index p in G with centers that have index p^3 in G, and $A_1, \ldots, A_m$ $(m \geq 0)$ are all of the subgroups (other than A) of index p^2 in G with centralizers that have index p^2 in G. Furthermore, if $n \geq 1$, then $m \geq p$.

   c) *If $|G : Z(G)| = p^i$ with $i > 4$ and if G possesses a subgroup, T, of index p in G with center that has index p^3 in G, then $\mathcal{CD}(G) = \{Z(T), A, A_1, \ldots, A_p, T\}$ is a quasi-antichain of width $p + 1$, with each A_i abelian.*

If p is the smallest prime divisor of $|G|$ and if $|G : Z(G)| > p^4$ and if G possesses a subgroup, T, of index p in G with center that has index p^3 in G, then $\mathcal{CD}(G) = \{Z(T), A, A_1, \ldots, A_p, T\}$ is a quasi-antichain of width $p+1$, with each A_i abelian.

d) *If $|G : Z(G)| = p^i$ with $i > 4$ and if G does not possess a subgroup, T, of index p in G with center that has index p^3 in G, then $\mathcal{CD}(G) = \{A\}$.*

If p is the smallest prime divisor of $|G|$ and if $|G : Z(G)| > p^4$ and if G does not possess a subgroup, T, of index p in G with center that has index p^3 in G, then $\mathcal{CD}(G) = \{A\}$.

Proof. Item 1 is clear.

To see item 2, note first that $|G : G||G : Z(G)| = |G : Z(G)| = p^2 \leq |G : T||G : C_G(T)|$ for any subgroup T of G with $Z(G) \leq T$, and so $G, Z(G) \in \mathcal{CD}(G)$. Note that $G/Z(G) \cong C_p \times C_p$, and so there are exactly $p+1$ subgroups H with $Z(G) < H < G$. For any such H, we have that $H = \langle Z(G), x\rangle$ for some $x \in H \setminus Z(G)$, and thus H is abelian. It follows that $\mathcal{CD}(G) = \{Z(G), A, A_1, \ldots, A_p, G\}$ is a quasi-antichain of width $p+1$, with each A_i abelian.

We prove both parts of item 3 simultaneously. Note that $|G : G||G : Z(G)| = |G : Z(G)| > p^2 = |G : A||G : A|$, and so $G, Z(G) \notin \mathcal{CD}(G)$. Note that $|G : A||G : A| = p^2 \leq |G : T||G : C_G(T)|$ for any subgroup T of G with $Z(G) \leq T$, and so $A \in \mathcal{CD}(G)$. Now A is a maximal subgroup of G and $G \notin \mathcal{CD}(G)$, and so A is the largest member of $\mathcal{CD}(G)$. But A is self-centralizing and therefore A is also the least member of $\mathcal{CD}(G)$. Hence $\mathcal{CD}(G) = \{A\}$.

We prove both parts of item 4.a simultaneously. Note that $|G : A||G : A| = p^4 > |G : Z(G)| = |G : G||G : Z(G)|$, and so $A \notin \mathcal{CD}(G)$. Since among the self-centralizing subgroups of G, A is one of maximum order, it follows that $\mathcal{CD}(G)$ does not contain any self-centralizing subgroups. Among the abelian subgroups of $\mathcal{CD}(G)$, let T be a maximal one. If $T \neq Z(G)$, then we have that $T < C_G(T) < G$, and furthermore we have that $|C_G(T) : T|$ cannot be a prime. This is true because if $|C_G(T) : T|$ is a prime, then $C_G(T)/T$ is cyclic, and hence $C_G(T)$ is abelian, and so T would not be maximal among the abelian subgroups in $\mathcal{CD}(G)$. It follows that $|G : T||G : C_G(T)| \geq p^4 > |G : G||G : Z(G)|$, a contradiction. Hence $T = Z(G)$ is the only abelian subgroup in $\mathcal{CD}(G)$. If $H \in \mathcal{CD}(G)$ is a subgroup of index p in

G with $C_G(H)$ a subgroup of index p^2 in G, then $|C_G(H) : Z(G)| < p^2$, and hence is prime. But then $C_G(H)$ is abelian, a contradiction. It follows that $\mathcal{CD}(G) = \{Z(G), G\}$.

To see item 4.b, note that $p^4 = |G : G||G : Z(G)| = |G : A||G : A|$. Since among the self-centralizing subgroups of G, A is one of maximum order, we have that any other self-centralizing subgroup A' of $\mathcal{CD}(G)$ will have that $p^4 = |G : A'||G : A'|$. If T is an abelian subgroup of $\mathcal{CD}(G)$ with $T < C_G(T) < G$ and $C_G(T)$ nonabelian, then as we saw in the proof of item 4.a, we have that $|G : T||G : C_G(T)| \geq p^4$. It follows then that $G, Z(G), A \in \mathcal{CD}(G)$. And hence

$$\{Z(G), Z(T_1), \ldots, Z(T_n), A, A_1, \ldots, A_m, T_1, \ldots, T_n, G\} \subseteq \mathcal{CD}(G)$$

where $T_1, \ldots, T_n$ ($n \geq 0$) are all of the subgroups of index p in G with centers that have index p^3 in G, and $A_1, \ldots, A_m$ ($m \geq 0$) are all of the subgroups (other than A) of index p^2 in G with centralizers that have index p^2 in G.

We also have that

$$\mathcal{CD}(G) \subseteq \{Z(G), Z(T_1), \ldots, Z(T_n), A, A_1, \ldots, A_m, T_1, \ldots, T_n, G\}.$$

This is true because, otherwise, if some $H \in \mathcal{CD}(G)$ has that $|G : H| = p$ and $|G : C_G(H)| = p^3$, but $Z(H) < C_G(H)$, then since $Z(H) = H \cap C_G(H) \in \mathcal{CD}(G)$, it would follow that $Z(H) = Z(G)$ and thus $H = G$, a contradiction.

If $n \geq 1$, then for some $T \in \{T_1, \ldots, T_n\}$, we have that $Z(T) < A < T$ in $\mathcal{CD}(G)$. This is true because, otherwise, we would have $Z(G) < A < G$ and $Z(G) < Z(T) < T < G$ as maximal chains in $\mathcal{CD}(G)$, contradicting the modularity of $\mathcal{CD}(G)$. We can apply item 2 in Theorem 2.1 to the group T, to conclude that there are p additional self-centralizing groups in $\mathcal{CD}(T)$, and since all of these are self-centralizing, they are also all in $\mathcal{CD}(G)$.

We prove both parts of item 4.c simultaneously. Note that $|G : G||G : Z(G)| > p^4 = |G : A||G : A|$, and so $G, Z(G) \notin \mathcal{CD}(G)$. As we saw before, if S is an abelian subgroup of $\mathcal{CD}(G)$ with $S < C_G(S) < G$ and $C_G(S)$ nonabelian, then $|G : S||G : C_G(S)| \geq p^4$. It follows that $A, T, Z(T) \in \mathcal{CD}(G)$, and since T is maximal in G, we have that T is the largest member of $\mathcal{CD}(G)$. And so applying Theorem 2.1 item 2 to the group T, we obtain that $\mathcal{CD}(G) = \{Z(T), A, A_1, \ldots, A_p, T\}$ is a quasi-antichain of width $p+1$, with each A_i abelian.

We prove both parts of item 4.d simultaneously. As we saw in the proof of item 4.c, $G, Z(G) \notin \mathcal{CD}(G)$ and $A \in \mathcal{CD}(G)$. But since there is no subgroup of index p in $\mathcal{CD}(G)$, we have that A is the largest member of $\mathcal{CD}(G)$, and since A is self-centralizing, $A = C_G(A)$ is the least member of $\mathcal{CD}(G)$, and we conclude that $\mathcal{CD}(G) = \{A\}$. $\square$

Items 3, 4.c, and 4.d in Theorem 2.1 have interesting consequences due to the uniqueness of the largest member of the Chermak-Delgado lattice. We collect them below as a corollary.

Corollary 2.2. *Let G be a finite group, let p be a prime, and among the self-centralizing subgroups of G, let A be one of maximum order.*

1) *If $|G : A| = p$ and $|G : Z(G)| = p^i$ with $i > 2$, then A is unique.*

   If p is the smallest prime divisor of $|G|$ and $|G : A| = p$ and $|G : Z(G)| > p^2$, then A is unique.

2) *If $|G : A| = p^2$ and $|G : Z(G)| = p^i$ with $i > 4$ and if G possesses a subgroup, T, of index p in G with center that has index p^3 in G, then T is unique.*

   If p is the smallest prime divisor of $|G|$ and if $|G : A| = p^2$ and $|G : Z(G)| > p^4$ and if G possesses a subgroup, T, of index p in G with center that has index p^3 in G, then T is unique.

3) *If $|G : A| = p^2$ and $|G : Z(G)| = p^i$ with $i > 4$ and if G does not possess a subgroup, T, of index p in G with center that has index p^3 in G, then A is unique.*

   If p is the smallest prime divisor of $|G|$ and if $|G : A| = p^2$ and $|G : Z(G)| > p^4$ and if G does not possess a subgroup, T, of index p in G with center that has index p^3 in G, then A is unique.

Examples for items 4.a and 4.b in Theorem 2.1 that are chains of length 1 and length 2 (and so in the case of item 4.b, $n = 0$ and $m = 0$), can be found in [3], see Corollary 2.2, Proposition 2.3, Corollary 2.5, and Proposition 2.6 there.

An example for item 4.b in Theorem 2.1 with $n = 1$ and $m = p$ will be given in Section 4.

Another example for item 4.b in Theorem 2.1 is found by taking G to be one of the two extraspecial p-groups of order p^5. $|Z(G)| = p$ (i.e. $|G : Z(G)| = p^4$) and G possesses a maximal self-centralizing elementary abelian subgroup of order p^3 (i.e. of index p^2 in G). The Chermak-Delgado lattice of extraspecial p-groups is known, see Example 2.8 in [6]. And so we have that $\mathcal{CD}(G)$ is isomorphic to the subgroup lattice of an elementary abelian p-group of order p^4. Using the notation of item 4.b in Theorem 2.1, we have that $n = p^3 + p^2 + p + 1$ and $m = (p^2 + 1)(p^2 + p + 1) - 1$.

Further examples of groups and CD lattices arising from Theorem 2.1 will be explored in the next sections.

3 Dicyclic groups

As a consequence of items 2 and 3 in Theorem 2.1, we classify the Chermak-Delgado lattice of $Dic_{4n}(A)$, where A is an abelian group of order $2n$.

Note that $\mathcal{CD}(Dic_{4n}(A)) = \{A\}$ if and only if $|Dic_{4n}(A) : Z(Dic_{4n}(A))| > 4$, which is equivalent to $|A^2| > 2$. If A is not an elementary abelian 2-group, this means that A is not a direct product of an elementary abelian 2-group and $\mathbb{Z}_4$.

Theorem 3.1. *Let A be an abelian group of order $2n$ and $Dic_{4n}(A)$ be the generalized dicyclic group induced by A. Then we have:*

a) *If A is not of type $\mathbb{Z}_2^m \times \mathbb{Z}_4$ with $m \in \mathbb{N}$, then $\mathcal{CD}(Dic_{4n}(A))$ is a chain of length 0, namely $\mathcal{CD}(Dic_{4n}(A)) = \{Dic_{4n}(A)\}$ for $\exp(A) = 2$, and $\mathcal{CD}(Dic_{4n}(A)) = \{A\}$ for $\exp(A) \neq 2$.*

b) *If A is of type $\mathbb{Z}_2^m \times \mathbb{Z}_4$ with $m \in \mathbb{N}$, then $\mathcal{CD}(Dic_{4n}(A))$ is a quasi-antichain of width 3, namely the lattice interval between $Z(Dic_{4n}(A))$ and $Dic_{4n}(A)$.*

As a corollary, we describe the Chermak-Delgado lattice of Dic_{4n}.

Corollary 3.2. *Under the previous notation, we have:*

a) *If $n \neq 2$, then $\mathcal{CD}(Dic_{4n})$ is a chain of length 0, namely $\mathcal{CD}(Dic_{4n}) = \{Dic_{4n}\}$ for $n = 1$, and $\mathcal{CD}(Dic_{4n}) = \{\langle a\rangle\}$ for $n \geq 3$.*

b) *If $n = 2$, then $\mathcal{CD}(Dic_{4n})$ is a quasi-antichain of width 3, namely the lattice interval between $Z(Dic_{4n})$ and Dic_{4n}.*

The following appears in [10].

Theorem 3.3. *Let G be a finite group which can be written as $G = AB$, where A and B are abelian subgroups of relatively prime orders and A is normal. Then*

$$m(G) = |A|^2 |C_B(A)|^2 \text{ and } \mathcal{CD}(G) = \{AC_B(A)\}.$$

Corollary 3.4. *Suppose A is a finite abelian group. There exists a non-abelian finite group G so that $\mathcal{CD}(G) = \{A\}$ if and only if $A \neq 1$, $A \not\cong \mathbb{Z}_2$, $A \not\cong \mathbb{Z}_4$, and $A \not\cong \mathbb{Z}_2 \times \mathbb{Z}_4$.*

Proof. $\rightarrow$ Suppose $\mathcal{CD}(G) = \{A\}$ and G is non-abelian. If $A = 1$, then $G = 1$ is abelian, a contradiction. Since A is self-centralizing and normal in G, we have that G/A embeds into $Aut(A)$. If $A \cong \mathbb{Z}_2$, then G/A embeds into $Aut(A) = 1$, which contradicts G being non-abelian. If $A \cong \mathbb{Z}_4$, then G/A embeds into $Aut(A)$ which has order 2, and so $|G| = 8$ and $CD(G)$ would not be a chain of length zero by item 2 in Theorem 2.1. Suppose $A \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Then G/A embeds into $Aut(A)$ which is a dihedral group of size 8. And so G is a 2-group. Now, $m^*(G) = 64$, and since $G \notin \mathcal{CD}(G)$ and G has a nontrivial center, it follows that $|G| = 16$ and $|Z(G)| = 2$. Note, however, that given $x \in Aut(A)$ of order 2, x must fix at least four elements of A. This is due to the structure of $Aut(A)$ when $A \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. And so $|Z(G)| > 2$, a contradiction.

$\leftarrow$ If $|A|$ is odd, then let $G = A \rtimes \langle x \rangle$ where x inverts each element of A. Then by item 3 in Theorem 2.1, $\mathcal{CD}(G) = \{A\}$. If $|A|$ is even, and A is not of type $\mathbb{Z}_2^m \times \mathbb{Z}_4$ with $m \geq 0$, and $\exp(A) \neq 2$, then $G = Dic_{4n}(A)$ has that $\mathcal{CD}(G) = \{A\}$ by Theorem 3.1. If $A \cong \mathbb{Z}_2^m$ with $m \geq 2$ or $A \cong \mathbb{Z}_2^m \times \mathbb{Z}_4$ with $m \geq 2$, then there exists $x \in Aut(A)$ of order 3. Let $G = A \rtimes \langle x \rangle$. One can apply Theorem 3.3 to get that $\mathcal{CD}(G) = \{A\}$. $\square$

Finally, we present a proposition also applicable to describing $\mathcal{CD}(Dic_{4n})$.

Proposition 3.5. *Let G be a finite group. If $G = HZ(G)$ for some subgroup H of G, then $\mathcal{CD}(G)$ and $\mathcal{CD}(H)$ are lattice isomorphic with $\mathcal{CD}(G) = \{XZ(G) \mid X \in \mathcal{CD}(H)\}$.*

Proof. Since $G = HZ(G)$, we have that $G/Z(G) \cong H/(Z(G) \cap H) = H/Z(H)$ since $Z(H) \leq Z(G)$. This natural isomorphism between $G/Z(G)$ and $H/Z(H)$ induces a lattice isomorphism between the subgroup intervals $[G : Z(G)]$ and $[H : Z(H)]$, and this lattice isomorphism restricts to a lattice isomorphism between $\mathcal{CD}(G)$ and $\mathcal{CD}(H)$. The result follows. $\square$

4 Metabelian p-groups of maximal class

By applying Lemmas 1.1, 1.2, 1.3 and Theorem 2.1, we obtain a classification of the Chermak-Delgado lattices of a metabelian p-groups of maximal class.

Theorem 4.1. *Suppose G is a metabelian p-group of order p^n and of maximal class.*

1) *If $n = 3$, then $\mathcal{CD}(G) = \{Z(G), A, A_1, \ldots, A_p, G\}$ is a quasi-antichain of width $p + 1$, with each A_i abelian.*

2) *If G possesses an abelian subgroup A so that $|G : A| = p$ and $n > 3$, then $\mathcal{CD}(G) = \{A\}$.*

3) *Suppose that G does not possess an abelian subgroup A so that $|G : A| = p$. Then we have that $n > 4$, and, furthermore:*

   a) *If $|G| = p^5$, then $\mathcal{CD}(G) = \{Z(G), Z(T), G', A_1, \ldots, A_p, T, G\}$, where $|T| = p^4$, $|G'| = |A_i| = p^3$ for each i, $|Z(T)| = p^2$, and $G', A_1, \ldots, A_p$ are all abelian and distinct. Also, none of $A_1, \ldots, A_p$ are normal in G.*

   b) *If $|G| > p^5$ and G possesses a subgroup T of index p so that $|T : Z(T)| = p^2$, then $\mathcal{CD}(G) = \{Z(T), G', A_1, \ldots, A_p, T\}$, where $|T| = p^{n-1}$, $|G'| = |A_i| = p^{n-2}$ for each i, $|Z(T)| = p^{n-3}$, and $G', A_1, \ldots, A_p$ are all abelian and distinct. Also, none of $A_1, \ldots, A_p$ are normal in G.*

   c) *If $|G| > p^5$ and G does not possess a subgroup T of index p so that $|T : Z(T)| = p^2$, then $\mathcal{CD}(G) = \{G'\}$.*

Proof. Items 1 and 2 follow directly from items 2 and 3 in Theorem 2.1. The fact that $n > 4$ in item 3 follows from Lemma 1.1. Most of Theorem 4.1 part 3 is a straight forward application of Lemma 1.2 and Theorem 2.1 taking $A = G'$. In item 3.a, the fact that there is a unique subgroup, T, of index p in G in $\mathcal{CD}(G)$ is a nontrivial result that follows from Lemma 1.3 which in turn relies on Lemma 4.5.4 in [14]. In items 3.a and 3.b, the fact that none of $A_1, \ldots, A_p$ are normal in G follows from Lemma 1.2. □

Using Theorem 4.1, we are able to give some necessary and sufficient conditions under which the Chermak-Delgado lattice of a metabelian p-group of maximal class is a chain of length 0.

Corollary 4.2. *Let G be a metabelian p-group of maximal class. Then $\mathcal{CD}(G)$ is a chain of length 0 if and only if either G possesses an abelian subgroup of index p and $|G : Z(G)| \neq p^2$, or G does not possess abelian subgroups of index p, $|G| > p^5$, and $|M : Z(M)| > p^2$ for any maximal subgroup M of G.*

We provide now a few remarks. The 2-groups of maximal class fall into 3 categories: the dihedral groups, the semidihedral groups, and the generalized quaternion groups (see e.g. Theorem 4.1 of [12], II), and each of these possess a cyclic subgroup of index 2. Thus, the CD lattices of these groups fall under the umbrella of items 2 and 3 in Theorem 2.1.

Xue, Lv, and Chen provide sufficient conditions for when p-groups of maximal class have CD lattices that fall under the umbrella of item 3 in Theorem 2.1 (see Theorem 2.9 of [17]):

Proposition 4.3. *Let G be a finite p-group of order p^n and of maximal class, where $p \geq 5$ and $n > 2p$. If $C_G(H) = Z(H)$ for every non-abelian subgroup H of G, then G possesses an abelian subgroup of index p.*

Note that the finite non-abelian groups satisfying the condition in Proposition 4.3 are called *CGZ-groups*. Xue, Lv, and Chen show that if G is a p-group of maximal class and G is a CGZ-group, then G is metabelian (see Proposition 2.6 of [17]).

We end with examples of metabelian p-groups of maximal class for each of the three cases in item 3 of Theorem 4.1. For the case a) such an example is presented in [17]:

$$G=\langle a,b,c,d \mid a^{p^2}=b^p=c^p=d^p=1, [b,a]=c, [c,a]=d, [c,b]=[d,b]=a^p, [d,a]=1\rangle,$$

where $p \geq 5$.

For case b) we present $SmallGroup(3^6, 99)$ from GAP:

$$G=\langle a,b,c,d,e,f \mid a^3=e^3=f^3=1, b^3=c^2d, c^3=e^2f, d^3=f^2, [b,a]=c, [c,a]=d, [d,a]=e,$$

$$[e,a]=[c,b]=f, [f,a]=[d,b]=[e,b]=[f,b]=[d,c]=[e,c]=[f,c]=[e,d]=[f,d]=[f,e]=1\rangle.$$

For case c) we present $SmallGroup(5^6, 651)$ from GAP:

$$G{=}\langle a,b,c,d,e,f \mid a^5{=}c^5{=}d^5{=}e^5{=}f^5{=}1, b^5{=}f^3, [b,a]{=}c, [c,b]{=}d, [c,a]{=}[d,b]{=}e,$$

$$[d,a]{=}[e,b]{=}f, [e,a]{=}[f,a]{=}[f,b]{=}[d,c]{=}[e,c]{=}[f,c]{=}[e,d]{=}[f,d]{=}[f,e]{=}1\rangle.$$

Acknowledgments. The authors are grateful to the reviewer for its remarks which improve the previous version of the paper.

References

[1] L. An, J.P. Brennan, H. Qu and E. Wilcox, *Chermak-Delgado lattice extension theorems*, Comm. Algebra **43** (2015), 2201-2213.

[2] B. Brewster and E. Wilcox, *Some groups with computable Chermak-Delgado lattices*, Bull. Aus. Math. Soc. **86** (2012), 29-40.

[3] B. Brewster, P. Hauck and E. Wilcox, *Groups whose Chermak-Delgado lattice is a chain*, J. Group Theory **17** (2014), 253-279.

[4] B. Brewster, P. Hauck and E. Wilcox, *Quasi-antichain Chermak-Delgado lattices of finite groups*, Archiv der Mathematik **103** (2014), 301-311.

[5] A. Chermak and A. Delgado, *A measuring argument for finite groups*, Proc. AMS **107** (1989), 907-914.

[6] G. Glauberman, *Centrally large subgroups of finite p-groups*, J. Algebra **300** (2006), 480-508.

[7] I.M. Isaacs, *Finite group theory*, Amer. Math. Soc., Providence, R.I., 2008.

[8] R. McCulloch, *Chermak-Delgado simple groups*, Comm. Algebra **45** (2017), 983-991.

[9] R. McCulloch, *Finite groups with a trivial Chermak-Delgado subgroup*, J. Group Theory **21** (2018), 449-461.

[10] R. McCulloch and M. Tărnăuceanu, *Two classes of finite groups whose Chermak-Delgado lattice is a chain of length zero*, Comm. Algebra **46** (2018), 3092-3096.

[11] R. Schmidt, *Subgroup lattices of groups*, de Gruyter Expositions in Mathematics 14, de Gruyter, Berlin, 1994.

[12] M. Suzuki, *Group theory*, I, II, Springer Verlag, Berlin, 1982, 1986.

[13] M. Tărnăuceanu, *The Chermak-Delgado lattice of ZM-groups*, Results Math. **72** (2017), 1849-1855.

[14] L.S. Vieira, *On p-adic fields and p-groups*, Ph.D. Thesis, University of Kentucky, 2017.

[15] E. Wilcox, *Exploring the Chermak-Delgado lattice*, Math. Magazine **89** (2016), 38-44.

[16] M. Xu, L. An and Q. Zhang, *Finite p-groups all of whose non-abelian proper subgroups are generated by two elements*, J. Algebra **319** (2008), 3603-3620.

[17] H. Xue, H. Lv and G. Chen, *On a special class of finite p-groups of maximal class*, Italian J. Pure Appl. Math. **33** (2014), 279-284.

[18] A. Morresi Zuccari, V. Russo, and C.M. Scoppola, *The Chermak-Delgado measure in finite p-groups*, J. Algebra **502** (2018), 262-276.

Ryan McCulloch
Assistant Professor of Mathematics
University of Bridgeport
Bridgeport, CT 06604
e-mail: `rmccullo@bridgeport.edu`

Marius Tărnăuceanu
Faculty of Mathematics
"Al.I. Cuza" University
Iaşi, Romania
e-mail: `tarnauc@uaic.ro`